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Optimal timing of price change with strategic customers under demand uncertainty: A real option approach

Lee, Y.[a], **Lee, J.P.**[b], **Kim, S.**[b,*]

[a]Department of Mechanical Engineering, University of Texas at San Antonio, San Antonio, United States of America (USA)
[b]College of Business Administration, Hongik University, Seoul, South Korea

ABSTRACT

This paper proposes a model to determine the optimal markdown timing for a company with strategic customer purchasing behaviour. Since strategic customers are aware of potential markdown under the posted pricing scheme, they may choose to wait longer to maximise their utilisation instead of buying a product and fulfilling an instant surplus. On the other hand, the seller can delay the markdown decision until it is proved to be profitable and hence has an option to determine the timing. In estimating the value of the markdown decision, the seller's option needs to be estimated. However, the value of the option is hard to be captured by the conventional net present value analysis. Under market uncertainty where potential customer demand evolves over time, the seller's revenue function is in the form of a stochastic dynamic programming model. Applying a real option approach, we investigate the optimal price path and propose the optimal markdown threshold. Given the markdown costs incurred, we find that the optimal discount timing for the firm is determined by a threshold policy. Furthermore, our results show that if future market becomes more uncertain, the seller needs to wait longer or delay the markdown decision. In addition, the optimal threshold of the markdown decreases exponentially in a declining market, which explains the early markdown policy of some consumer product companies.

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**Corresponding author:*
sbkim@hongik.ac.kr
(Kim, S.)

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1. Introduction

Demand management becomes the basis for the decision making of the firms; from production planning to inventory management [1, 20]. Pricing policies are frequently used tools when firms manage their demand [20]. The pricing policies of a firm are often complex and diverse depending on the business environment in which the company lies [1, 5, 6, 14, 20]. In the fashion industry, for instance, simple markdown pricing is widely used to sell out remaining stock after the regular sales season [21]. Some customers may choose to wait and purchase the product later at the markdown price rather than buying it right away. On the other hand, airline companies continuously mark up the prices of tickets upon departure. When looking for an airline ticket, customers can expect an increase in prices if they delay their purchase. Therefore, understanding customer purchasing behaviour is critical for the firm to make pricing decisions.

Strategic customers in the operations management literature are defined as those who are aware of the firm's dynamic pricing policies and make inter-temporal purchasing decisions [20]. Since such customers are conscious of potential changes in prices at a later point in time, they are being strategic rather than myopic. Instead of buying a product and fulfilling an instant surplus, they strategically wait for a future price markdown and thereby seek to maximise their utilisation [5, 6, 18]. As such, strategic customers have become a substitute for myopic customers who simply make a buying decision if the price is lower than their valuation [5, 6, 20]. Therefore, firms must comprehend the strategic behaviour of customers and find an optimal pricing scheme based on it to maximise revenue.

In response to the strategic behaviour of customers, the firm's decisions are generally twofold: the timing of price changes and the availability of the product [3, 20]. The firm sells a product for a duration of time, after which it may decide to change the price at a certain point in time. Limited supply could also be used as a marketing strategy to increase the sense of urgency among customers. Therefore, customers in the market choose either to purchase a product at its current price or to revisit it after the price goes down, considering not only the timing of the markdown but also the possibility of sellouts.

In this paper, we investigate the markdown decision of a monopolist who wishes to maximise expected revenues in the presence of strategic customers. Our model captures several important properties of the market environment for consumer products. First, the seller commits to a fixed path of two prices: it may sell a product for a duration of time, after which it may decide to change the price. The markdown decision can be made no more than once over the sales horizon and is hence irreversible. Second, customers show strategic purchasing behaviour towards firms: even if the valuation of the product exceeds the price of the product during the first part of the sales horizon, customers may not simply purchase it. Instead, their decision to purchase is based on the valuation that exceeds a certain level, thus following a threshold policy. Third, potential customer demand is stochastic. In particular, the market size follows a geometric Brownian motion that evolves dynamically over time.

We consider a seller's problem on deciding the optimal price path and the timing of a markdown under demand uncertainty. Specifically, we present a stochastic dynamic programming model where the seller has a single opportunity to discount the price of the product at a sunk cost. Customers in the market are aware of a potential markdown and the likelihood of a sellout. Based on customers' valuation in regard to the two prices, the value of the firm is expressed as a stream of expected revenues. Solving the problem using a real option approach, we show that the seller's optimal markdown timing decision is based on the threshold policy. To the best of our knowledge, this is one of the first studies that considers a posted pricing scheme under market uncertainty.

The remainder of the paper is organised as follows. Section 2 outlines previous related works to summarise extant research. Section 3 proposes a revenue maximisation model, and Section 4 continues with the topic by analysing the solution of the model. Finally, we discuss broader findings, conclusions, and potential future research opportunities in Section 5.

2. Literature review

Strategic customers and firms' pricing policy problems have become an increasingly productive research area. Among others, studies regarding customer purchasing behavior are thoroughly reviewed by Shen and Su [20]. Most of the papers in the literature consider two important elements in modelling strategic customer behaviour. The first is the arrival process of customers. Whether customers preexisted in the market [7, 13, 18] or sequentially arrived in the market [3, 14, 21, 23] is a question based on this premise. The second is the decision making of the customers and how the decision making ultimately forms an equilibrium. Regardless of the market size, the decision making of an individual customer makes an impact on the dynamics of the market to some extent. For instance, when many customers purchase goods in the early stages, the product may run out of stock for some of those who initially decided to delay the purchase [13]. Sometimes customers may have to purchase the goods at an even higher price if the firm or seller chooses to adopt a markup pricing policy [24]. A strategic customer tries to make an optimal decision, foreseeing these situations, and this process may, in turn, comprise an equilibrium in decision making. The seller, on the other hand, sets his or her pricing policy based on this equilibrium in an effort to maximise profit. Therefore, this game-theoretic relationship with conflicting interests between the seller and the strategic buyers necessarily leads to a highly complex model in many studies.

There are two types of simplifications to deal with the complexity in the modelling. Firstly, the time of the price change is often fixed. A specific number of periods are presumed, and static pricing is maintained for the duration of the periods. In other words, the analysis of optimal pricing is based on the definite number of periods in which a fixed price is offered, rather than finding the price changing period one by one [5-7, 13]. The second case is the size of the market: in applying a game theory approach, a small market size is assumed. In this situation, a customer predicts the decision making of other consumers to make his or her optimal decision and an equilibrium of strategic purchases is achieved. Aviv and Pazgal [3] found that a firm's benefits from price differentiation may decrease as customers become more strategic, and hence optimal pricing policies may result in potential revenue losses in the presence of strategic customers.

Customers' purchasing decisions depend on the interaction among the pricing policy, availability, customer valuations, remaining time, and so forth. Under the posted pricing, for instance, where the seller announces its price path in advance, the availability of the product or the possibility to purchase it later in time will be the major concern for the customer [2, 7, 13]. As Dasu and Tong [7] specifically pointed out, the seller's dynamic pricing decision is meaningful only if customers are aware of the stock-out possibility, while the impact of the perception on strategic customer behaviour is different in heterogeneous customer valuations [23]. In many studies, a two-period posted pricing scheme has been used due to its simplicity and applicability, although the seller can still make a price change at any time [4, 7, 13, 15]. Dasu and Tong [7], in particular, found that the approximation close to the maximum revenue can be achieved by two or three pricing changes. In this study, our model will also be based on the two-period posted pricing in continuous time periods to find the optimal timing of price change, while the availability of the item is limited after the markdown.

As for the firm's point of view, on the other hand, market size is the main source of uncertainty. Given the price and the timing of the price change, the firm's revenue must be significantly different depending on changes in demand at the moment. Under market uncertainty, the seller can either make an immediate price change or intentionally delay the decision to observe the actual demand movement. This situation is very common in many operational practices: companies have an opportunity to invest but they can still wait for new information. In other words, a firm with the ability to postpone a decision has the option, not the obligation, to exercise it – making it analogous to holding a financial call option. Since first proposed by Pindyck [19], McDonald and Siegel [17], Dixit and Pindyck [8] and others, this real option approach has been widely borrowed in the areas of marketing and operations management because it helps us to better understand the true value of the investment opportunity.

Adopting the real option concept is not completely new in revenue management literature. In numerous papers, the dynamic pricing decision is determined by considering the option value of unsold products [11, 16]. Since this option value decreases towards the end of the time horizon, the optimal price path also decreases over time. In another paper, Gallego and Sahin [10] used the real option approach to model uncertain customer valuations. In this paper, however, we assume that potential customer demand evolves over time and follows the geometric Brownian motion (GBM). Assuming the known distribution on customer valuations and the level of availability, we explore the optimal markdown timing problem based on the net present value of the seller's expected revenue. To the best of our knowledge, in the literature on strategic customers, there are only a handful of studies that deal with the optimal timing of price change, and yet fewer still that at the same time address optimal pricing with strategic customers under market uncertainty.

3. Model description

In this paper, we consider a monopolistic firm that sells a single item to potential customers over two periods. The firm wants to maximise its net present value of expected revenue. Below, we explain further assumptions before building our model.

Assumption 1. *The monopolistic firm follows a two-period markdown pricing scheme and commits to the price path in both phases.*

Assumption 2. *The original (p_o) and markdown prices (p_l) are pre-announced and the in-stock probability (π) in the second period is also given information.*

Assumption 3. *Customers are strategic rather than myopic and are aware of markdowns and possibilities of stock-outs.*

Assumption 4. *The distribution of customer valuations ($G(\cdot)$) is known.*

3.1 Valuation of a strategic customer

Suppose that there is a monopolist who has a sufficiently large number of an item. Until time T, the item is initially sold at price p_o, and after T the item is sold at the markdown price p_l ($\leq p_o$). The two prices, p_o and p_l, are pre-announced. Customer demand follows a geometric Brownian motion (GBM), and each customer is supposed to purchase only one unit of the item. When the seller offers a markdown price, we assume that the seller can control the level of product availability, π, to induce scarcity. In other words, in the second period, the in-stock probability decided by the seller will be set to $\pi \leq 1$. Controlling the availability of services or items of different classes is prevalent in revenue management [24] and inducing a level of scarcity is also one of the most common strategies in marketing [9, 22].

Let U denote the customer's surplus. Then the utilisation of the customer who purchases the item right now is as follows:

$$U_o = V - p_o \tag{1}$$

where U_o is the surplus of the customer and p_o is the current price of the product.

Similarly, the utilisation of the customer who decides to wait for the discount is as follows:

$$U_l = \pi(V - p_l) + \theta \tag{2}$$

where V is the customer's valuation of the product, p_o is the current price of the product, p_l is the future price of the product, and π is the service level of the product at the lower price p_l. Thus, the stock-out probability is $1 - \pi$. θ stands for the customer's preference for risk; $\theta < 0$ indicates risk-averse, $\theta > 0$ risk-taking, and $\theta = 0$ risk-neutral attitude. Furthermore, we assume that the customers are either risk-averse or risk-neutral, which is a prevalent assumption made by many researchers [13, 15].

In this setting, the strategic customers decide to purchase in the first period if their valuation is greater than or equal to the threshold value. The purchasing decision of the strategic customer is determined by the following lemma.

Lemma 1. *The threshold of a strategic customer's valuation is given by*:

$$\tau = \frac{p_o - \pi p_l + \theta}{1 - \pi} \tag{3}$$

Proof. The two choices, purchasing right now or waiting for a discount, generate the same surplus when $U_o = U_l$. Solving the equation, a strategic customer will have the following threshold. That is,

$$V - p_o = \pi(V - p_l) + \theta. \tag{4}$$

This would finish the proof.

If $\tau > 1$, no customers would buy in the first period. Furthermore, we assume that customers do not purchase if the utilisation is less than zero without loss of generality. That means, τ is not less than p_o. Therefore, it is sufficient to consider only the case where the threshold is between the first period's pricing and one. That is,

$$p_o \leq \tau \leq 1 \tag{5}$$

This also decides the upper and the lower bounds of θ accordingly.

Following the literature, potential customer demand is assumed to be a multiplication of the customer value function and the time-varying potential demand. That is, the demand function is given by

$$Q = \bar{G}(V)X = \{1 - \bar{G}(V)\}X \tag{6}$$

where $\bar{G}(V)$ is a known distribution function of the product at customer valuation V , and X is the multiplicative demand shock process. This may be thought as demand in which the product has a unit price.

In the first period, a strategic customer would purchase the product, if $U_o \geq U_l$ and $U_o \geq 0$. Therefore, from the conditions,

$$V - p_o \geq \pi(V - p_l) + \theta \text{ and } V \geq p_o \tag{7}$$

we have

$$V \geq \tau. \tag{8}$$

Since $V \geq \tau$ for the customers purchasing in the first period, the current demand function is

$$Q_o = \{1 - G(\tau)\}X = \left\{1 - G\left(\frac{p_o - \pi p_l + \theta}{1 - \pi}\right)\right\}X. \tag{9}$$

On the other hand, a proportion of customers would wait and purchase later at a lower price, if $U_l \geq U_o$ and $U_l \geq 0$. The valuation of such customers is as follows:

$$p_l \leq V \leq \tau. \tag{10}$$

Hence, the demand function of the customers who come back later in the second period to purchase will be:

$$Q_s = \{G(\tau) - G(p_l)\}X = \left\{G\left(\frac{p_o - \pi p_l + \theta}{1 - \pi}\right) - G(p_l)\right\}X. \tag{11}$$

Finally, when the product starts being sold at a markdown price p_l, any customer whose valuation is at least greater than the price would purchase it. Namely, the demand function will be:

$$Q_l = \{1 - G(p_l)\}X. \tag{12}$$

Without loss of generality, let the valuation of customers, V, be uniformly distributed over [0,1]. Then the demand for each case is given as follows:

$$Q_o = \left\{-\frac{p_o^2}{1 - \pi} + \frac{\pi p_o p_l}{1 - \pi} + \frac{(1 - \pi + \theta)p_o}{1 - \pi}\right\}X \tag{13}$$

$$Q_s = \left(-\frac{\pi p_l^2}{1 - \pi} + \frac{\pi p_o p_l}{1 - \pi} + \frac{\pi \theta p_l}{1 - \pi}\right)X \tag{14}$$

$$Q_l = (1 - p_l)X. \tag{15}$$

3.2 Customer demand

In this paper, we use a geometric Brownian motion (GBM) to formulate the multi-plicative demand shock X at time t. That means the relative change in demand, dX_t/X_t, within a short time interval, $[t, t + dt]$, can vary with time t. The dynamics of demand are represented by the following formula:

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \tag{16}$$

where μ is the growth rate or drift rate in demand, σ is the volatility of the process, and W_t is a standard Wiener process. If $\mu > 0$, market size is increasing over time. If $\mu < 0$, market size is decreasing.

This continuous random variable X_t is said to have lognormal distribution because the integral of Eq. 16 gives the following demand function (see Appendix A for the derivation):

$$X_t = X_0 e^{(\mu - \sigma^2/2)t + \sigma W_t}, \tag{17}$$

where X_0 is the initial demand. While the bell-shaped pattern of demand is expected by Eq. 17, the realisation of demand will substantially deviate from it, depending on the market volatility, as shown in Fig. 1.

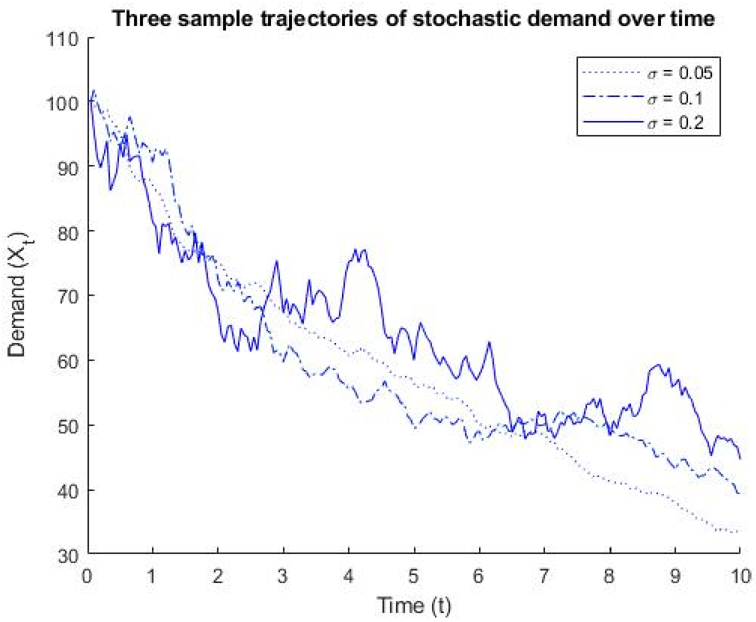

Fig. 1 Sample path of potential demand from the geometric Brownian motion in a decreasing market.

Note that the three sample paths in Fig. 1 are drawn from Eq. 16 with a mean drift rate of $\mu = -0.1$ and three standard deviations of $\sigma = 0.05, 0.1$, and 0.2. As shown in the figure, the sample path with a larger standard deviation tends to fluctuate significantly, while all three trajectories have a decreasing trend in common due to the negative mean drift rate.

3.3 Optimal timing of price discount

Next, we consider the optimal timing problem conditioned on the customer's purchasing strategy. We develop a model for an optimal discount timing decision using a real option model. In practice, the company has an "option" to delay the discount and hence needs to determine when the price should be discounted. After markdown, the company would make revenue $\Omega(X_T)$, with an irreversible sunk cost K being incurred from sales promotion, inventory management, and so forth.

Herein we formulate the value function of the firm with an opportunity of the discount timing T. When the product is sold at the original price p_o, a proportion of customers, Q_o, whose valuation is far higher than p_o, or greater than τ , will decide to purchase the item. A group of strategic customers, Q_s, whose valuation is between τ and p_o would like to wait and see if the price is marked down. Once the firm decides to discount the original price to the markdown price, p_l, they come back to purchase the product but only π of them will be able to get one. We assume that such demand is instantaneous, meaning that customer demand accumulated up to time T will be realised at time T [3]. From time T, any customers whose valuation is at least higher than p_l would like to purchase the product but, again, only π of them would get one.

We begin with the value function of the firm for the optimal discount timing problem. The value of the firm, $F(X)$, is the stream of revenue, which consists of three cases stated earlier. We use dynamic programming, stipulating an exogenous discount rate r. Then $F(X)$ is the expected present value

$$F(X) = \max_T E\left[\int_0^T e^{-rt} p_o(1-\tau)X_t dt + e^{-rT}\int_0^T \pi p_l(\tau - p_l)X_t dt + e^{-rT}\Omega'(X_T)\right] \tag{18}$$

where

$$\Omega'(X_T) = E\left[\int_T^\infty \pi p_l(1-p_l)X_t dt - K\right]. \tag{19}$$

Since the demand of strategic customers will be realised at time T, we rearrange the formula so that the revenue is included in the terminal payoff $\Omega(X_T)$. Then

$$(X) = \max_{T} E\left[\int_0^T e^{-rt} p_o(1-\tau)X_t dt + e^{-rT}\Omega(X_T)\right] \tag{20}$$

where

$$\Omega(X_T) = E\left[\int_0^T \pi p_l(\tau - p_l)X_t dt + \int_T^\infty \pi e^{-r(t-T)}(1-p_l)X_t dt - K\right] \tag{21}$$

$$= E\left[\int_0^T \pi p_l(\tau - p_l)X_t dt\right] + E\left[+\int_T^\infty \pi e^{-r(t-T)}(1-p_l)X_t dt\right] - K \tag{22}$$

$$= \pi p_l(\tau - p_l)E\left[\int_0^T X_t dt\right] + \pi(1-p_l)E\left[\int_T^\infty e^{-r(t-T)}X_t dt\right] - K \tag{23}$$

Since $E\left[\int_0^T X_t dt\right] = (e^{\mu T} - 1)X_0/\mu$ (See Appendix A) and $E\left[\int_T^\infty e^{-r(t-T)}X_t dt\right] = X_T/(r-\mu)$, we finally have:

$$\Omega(X_T) = \pi p_l(\tau - p_l)(e^{\mu T} - 1)\frac{X_0}{\mu} + \pi(1-p_l)\frac{X_T}{r-\mu} - K \tag{24}$$

Substituting τ in Eq. 3 into the formula, the value function of the firm is summarised as follows.

Proposition 1. *The value function of the firm for optimal discount timing T is given by:*

$$F(X) = \max_{T} E\left[\int_0^T \frac{e^{-rt}}{1-\pi}\{-p_o^2 + \pi p_o p_l + (1-\pi+\theta)p_o\}X_t dt + e^{-rT}\Omega(X_T)\right] \tag{25}$$

where

$$\Omega(X_T) = \frac{\pi}{1-\pi}(-p_o^2 + p_o p_l + \theta p_l)(e^{\mu T} - 1)\frac{X_0}{\mu} + \pi(1-p_l)\frac{X_T}{r-\mu} - K \tag{26}$$

By solving this stochastic dynamic programming problem, we can obtain the optimal timing for a markdown. The option-like approach shown in [17] and [19] is used to solve the dynamic stochastic problem. As the potential demand X evolves stochastically, the optimal strategy is to exercise (markdown) so that the value is at least greater than the critical value X^*. A firm's optimal markdown timing solution is represented in the following proposition.

Proposition 2. *A company considering markdown of the retail price will have a value function as follows:*

$$F(X) = \begin{cases} \alpha X^\beta + \dfrac{p_o(1-\tau)}{r-\mu}X & if\ X < X^* \\ \pi p_l(\tau - p_l)(e^{\mu T} - 1)\dfrac{X_0}{\mu} + \pi(1-p_l)\dfrac{X}{r-\mu} - K & if\ X \geq X^* \end{cases} \tag{27}$$

where

$$X^* = \frac{\beta}{1-\beta} \cdot \frac{r-\mu}{\pi(1-p_l) - p_o(1-\tau)}\left\{K - \pi p_l(\tau - p_l)(e^{\mu T} - 1)\frac{X_0}{\mu}\right\} \tag{28}$$

$$\alpha = \frac{\pi(1-p_l) - p_o(1-\tau)}{r-\mu} \cdot \frac{(X^*)^{1-\beta}}{\beta} \tag{29}$$

$$\text{and } \beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \tag{30}$$

Proof. Let T denote the timing at which the firm discounts the original price of the item. As described earlier, the firm makes revenue flows of $p_o(1-\tau)X$ before the markdown. At time T, the firm would make revenue flow $\Omega(X_T)$ with the irreversible sunk cost K. Therefore, as shown in [8], the Bellman equation in the continuation region, where values of X are not optimal to markdown, is given by:

$$[rF - p_o(1-\tau)X]dt = E[dF], \tag{31}$$

which implies that over a time interval dt, the total expected return on the markdown opportunity is equal to its expected rate of capital appreciation.

Applying Ito's lemma, we have

$$dF = F'dX + \frac{1}{2}F''(dX)^2, \tag{32}$$

where $F' = dF/dX$ and $F'' = d^2F/dX^2$.

Substituting Eq. 16 and dividing through by dt, we have the following Bellman equation (see Appendix B for proof):

$$\mu XF' + \frac{1}{2}\sigma^2 X^2 F'' - rF + p_o(1-\tau)X = 0 \tag{33}$$

To ensure the existence of the optimal solution, we assume that $\mu < r$. The differential equation $F(X)$ must satisfy the following three boundary conditions:

$$F(0) = 0 \tag{34}$$

$$F(X^*) = \pi p_l(\tau - p_l)(e^{\mu T} - 1)\frac{X_0}{\mu} + \pi(1-p_l)\frac{X^*}{r-\mu} - K \tag{35}$$

$$F'(X^*) = \frac{\pi(1-p_l)}{r-\mu} \tag{36}$$

Eq. 34 holds based on the observation that it will stay zero if the stochastic process X goes to zero. The other two equations are to impose continuity and smoothness at the critical point X^*, the potential demand at which it is optimal to discount. Eq. 35 is the value-matching condition, indicating the revenue the firm makes upon markdown. Eq. 36 is the smooth-pasting condition at the point.

Therefore, the solution of the differential Eq. 33 must take the form

$$F(X) = \alpha X^\beta + \frac{p_o(1-\tau)}{r-\mu}X, \tag{37}$$

where α is a constant to be determined and β is one of the solutions of the following quadrature equation:

$$\frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - r = 0. \tag{38}$$

Solving the equation and take

$$\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1 \tag{39}$$

to ensure the boundary condition.

From the smooth pasting and the value-matching conditions, we have

$$F(X^*) = \alpha(X^*)^\beta + \frac{p_o(1-\tau)}{r-\mu}X^* = \pi p_l(\tau - p_l)(e^{\mu T} - 1)\frac{X_0}{\mu} + \pi(1-p_l)\frac{X^*}{r-\mu} - K \tag{40}$$

and

$$F'(X^*) = \alpha\beta(X^*)^{\beta-1} + \frac{p_o(1-\tau)}{r-\mu} = \frac{\pi(1-p_l)}{r-\mu}. \tag{41}$$

Solving these equations results in:

$$X^* = \frac{\beta}{1-\beta}\cdot\frac{r-\mu}{\pi(1-p_l) - p_o(1-\tau)}\left\{K - \pi p_l(\tau - p_l)(e^{\mu T} - 1)\frac{X_0}{\mu}\right\} \tag{42}$$

$$\alpha = \frac{\pi(1-p_l) - p_o(1-\tau)}{r-\mu}\cdot\frac{(X^*)^{1-\beta}}{\beta} \tag{43}$$

$$\text{and } \beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}. \tag{44}$$

Herein, only if $K \geq \pi p_l(\tau - p_l)(e^{\mu T} - 1)X_0/\mu$ and $\pi(1-p_l) \geq p_o(1-\tau)$, we have a positive threshold $X^* \geq 0$. This result leads to Proposition 3.

Again, the threshold X^* determines the optimal markdown timing for a firm. When the actual customer demand of the firm at time t is lower than the threshold X^*, it is beneficial to sell the product at the original retail price p_o, making the revenue stream of $p_o(1-\tau)/(r-\mu)$ as well as giving the flexibility that the firm can hold for the price markdown, measured by αX^{β}. On the other hand, when the actual demand is greater than the threshold X^*, the firm will decide to discount the price and take the benefit of markdown $\pi p_l(\tau - p_l)(e^{\mu T}-1)\frac{X_0}{\mu} + \frac{\pi(1-p_l)}{r-\mu}X$ by spending investment cost K.

4. Analysis and discussion

This section explains some of the important characteristics for optimal markdown approaches suggested earlier. First, the following proposition illustrates that there exists a positive threshold for the firm at any given time t under specific conditions for K and p_l.

Proposition 3. *Let T^* denote the optimal timing of markdown to maximise the firm value. Then the optimal markdown time is finite $T^* < \infty$ [12], and the first epoch that demand exceeds the threshold is estimated at the following time:*

$$T^* = \inf\{t \geq 0 \mid X_t \geq X^*\}, \tag{45}$$

where there exists a positive threshold $X^* = \frac{\beta}{1-\beta} \cdot \frac{r-\mu}{\pi(1-p_l)-p_o(1-\tau)}\left\{K - \pi p_l(\tau - p_l)(e^{\mu T}-1)\frac{X_0}{\mu}\right\}$ at time t if $K \geq \pi p_l(\tau - p_l)(e^{\mu T}-1)X_0/\mu$ and $p_l \leq 1 - p_o(1-\tau)/\pi$.

Proof. By Proposition 2.

Note that we assume a decreasing market size ($\mu < 0$). As time t increases, therefore, we can observe that the threshold X^* decreases exponentially, while the minimum value for the fixed cost K that is required for this approach to be feasible increases exponentially before hitting the lower bound $\underline{K}$ as shown in the following proposition.

Proposition 4. *As $t \to \infty$, we have a threshold $X^* \to 0$ and the lower bound of the fixed cost $\underline{K} \to -\pi p_l(\tau - p_l)X_0/\mu$.*

Again, as the threshold X^* for the markdown decreases exponentially, a firm is likely to decide on a price discount in the relatively early stages. It also indicates that no significant revenue is expected after a certain amount of time because of the reduction in customer demand, and hence the firm no longer needs to invest more in later stages under this approach.

As the optimal timing X^* is represented by some exogenous factors, we exploit the impact of the parameters on the threshold.

Proposition 5. *The optimal timing threshold increases with respect to demand volatility. That is,*

$$\frac{\partial X^*}{\partial \sigma} > 0 \tag{46}$$

Proof. Noting the fractional value ≥ 0, we take the derivative of β from Eq. 16 with respect to the demand volatility, σ. We know $\beta > 11$ is one of the solutions to the following quadratic function $Q(\beta) = 0$ (the other solution is $\beta < 0$), where

$$Q(\beta) = \frac{1}{2}\sigma^2\beta(\beta - 1) + \mu\beta - r. \tag{47}$$

Taking a derivative of the equation, we have

$$\frac{\partial Q}{\partial \beta}\frac{\partial \beta}{\partial \sigma} + \frac{\partial Q}{\partial \sigma} = 0. \tag{48}$$

Since $\partial Q/\partial \beta > 0$ and $\partial Q/\partial \sigma > 0$, we have

$$\frac{\partial \beta}{\partial \sigma} < 0. \tag{49}$$

Furthermore,

$$\frac{\partial}{\partial\beta}\left(\frac{\beta}{\beta-1}\right)=-\frac{1}{(\beta-1)^2}<0. \tag{50}$$

Finally, the derivative of the optimal threshold X^* with respect to σ is,

$$\frac{\partial X^*}{\partial\sigma}=\frac{\partial\beta}{\partial\sigma}\frac{\partial}{\partial\beta}\left(\frac{\beta}{\beta-1}\right)\frac{r-\mu}{\pi(1-p_l)-p_o(1-\tau)}\left\{K-\pi p_l(\tau-p_l)(e^{\mu T}-1)\frac{X_0}{\mu}\right\}>0 \tag{51}$$

This proposition indicates that the optimal markdown threshold increases as the variation in demand increases. Simply put, it is more beneficial for the firm to wait and delay the markdown, thereby avoiding the risk of making the instantaneous decision when the market is highly uncertain. The firm is willing to make the markdown decision, only when excessive revenue is expected where the amount of uncertainty regarding future demand is larger.

5. Conclusion

In this paper, the optimal pricing policy of a monopolistic firm is investigated with strategic customer behaviour. When customers strategically wait for a discount, the monopolist has an option to offer a markdown to maximise its revenue. Assuming that the underlying customer demand is stochastic, evolving dynamically over time, we develop a value function for the firm to find the optimal time for the discount. Using a real option approach, the stochastic dynamic programming model is solved. Given the fixed cost of the markdown, service level, and a known discounted price, the optimal policy for the firm is to follow the threshold policy. The seller maximises its revenue by discounting the price of the product when the potential customer demand is greater than the threshold value.

The contribution of this paper is as follows: Considering the optimal markdown decision for a monopolistic seller with strategic customers, we address the gap in other literature on these customers with problems under market uncertainty. A stochastic dynamic optimisation model is proposed to find the optimal markdown strategy of the seller. A real option approach is applied to obtain a closed-form solution of the firm's demand threshold. The analysis of the optimal timing reveals the relationship between the degree of market uncertainty and the markdown decision-making.

Although the optimal threshold policy is found, careful interpretations of the result are needed. First, customers are aware of potential markdowns while the discounted price is known. The seller may not exercise the option to markdown if the potential demand never exceeds the threshold. Second, we found that there is an exponential decrease in the threshold value in a declining market, which justifies the early markdown in some industries. On the other hand, the optimal markdown threshold increases as the variation in demand increases. This indicates that a firm needs to avoid the risk of committing markdown pricing too early when the market is highly uncertain. The company's manufacturing and production planning must be aligned with this strategic decision on the markdown timing.

There are many challenges involved in the proposed study for future research. Discussion over potential demand is recommended. Further investigation on the posted pricing scheme of demand diffusion can be developed where the new product gets adopted in the population over time. Another potential area of research would be the prediction of strategic customer demand by applying data-driven approaches, such as meta-heuristics and machine learning algorithms. Finally, an interesting extension would be to implement the proposed framework on real-world problems to demonstrate the practical implications of our model.

Conflict of interests

The authors thank the editor and two reviewers for their constructive comments, which helped us to improve this paper. The authors declare that there is no conflict of interests regarding the publication of this paper.

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Appendix A: Proof of Proposition 1

Since $E\left[\int_0^T X_t\, dt\right] = \int_0^T E\,[X_t]dt$ by Fubini's Theorem, we first apply Ito's lemma to $d\ln X_t$, to find $E[X_t]$:

$$d\ln X_t = \frac{1}{X_t}dX_t - \frac{1}{2}\frac{1}{X_t^2}(dX_t)^2 \tag{52}$$

$$= \frac{1}{X_t}(\mu X_t + \sigma X_t dz) - \frac{1}{2}\frac{1}{X_t^2}(X_t^2\sigma^2 z^2) \tag{53}$$

$$= \mu\, dt + \sigma\, dz - \frac{1}{2}\sigma^2\, dt \tag{54}$$

After integrating and applying the fundamental theorem of calculus, we obtain:

$$\ln X_t - \ln X_0 = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t \tag{55}$$

$$X_t = X_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t} \tag{56}$$

The general form of expectation for Gaussian random variable is $E[e^X] = E\left[e^{\mu+\frac{1}{2}\sigma^2}\right]$, where X has the law of a normal random variable with mean μ and variance σ^2. Since we know the standard Brownian motion $W_t \sim N(0,t)$, taking expectation on both sides yields the following [9]:

$$E[X_t] = X_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t} E[e^{\sigma W_t}] \tag{57}$$

$$= X_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t} e^{0+\frac{1}{2}\sigma^2 t} \tag{58}$$

$$= X_0 e^{\mu t} \tag{59}$$

Finally taking integral produces the following results:

$$\int_0^T E\,[Xt]dt = \int_0^T X_0 e^{\mu t} dt = \frac{X_0}{\mu}(e^{\mu T} - 1) \tag{60}$$

Appendix B: Proof of Theorem 1

Substituting Eq. 16 into Eq. 32, we have the following equation:

$$dF = F'(\mu X\, dt + \sigma X\, dz) + \frac{1}{2}F''(\mu X dt + \sigma X\, dW)^2 \tag{61}$$

$$= \mu X F' dt + \sigma X\, F' dW + \frac{1}{2}\mu^2 X^2 F''(dt)^2 + \mu\sigma X^2 F''(dt)(dW) + \frac{1}{2}\sigma^2 X^2 F''(dW)^2 \tag{62}$$

Taking expectations on both sides to apply some properties of GBM and discarding all terms involving dt to a power higher than 1, we have

$$\mathrm{E}[dF] = \left[\mu X F' + \frac{1}{2}F''\sigma^2 X^2\right]\mathrm{dt} = [rF - p_o(1-\tau)]dt. \tag{63}$$

Note that the term $(dt)(dW)$ has magnitude $(dt)^{3/2}$, $E[dW] = 0$, $E[(dW)^2] = dt$, and $E[dt] = 0$. After dividing through by dt, we have the Bellman Eq. 33.